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Pós-Graduação em Matemática

Functional Analysis Exercises

1 Normed vector Spaces

1.1 – Let E be a normed space. Show that

$$\big|\|u\| - \|v\|\big| \leq \|u - v\|, \ \forall u, v \in E.$$

Conclude that the norm is a continuous function.

1.2 – Let E be a normed space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. Prove that if $x_n \to x$ in E, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$.

1.3 – A series $\sum_{n=1}^{\infty} x_n$ in a normed vector space E is said to be **absolutely convergent** if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent.

a) If E is a Banach space, prove if a series is absolutely convergent series then it is convergent.

b) Suppose E is a Banach space. If there exists $(M_n)_{n\in\mathbb{N}} \subset [0, +\infty)$ such that

$$\|x_n\| \leq M_n, \ \forall n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} M_n < +\infty,$$

then the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent in E. This is known as **Weierstrass M-test**.

1.4 – Let E a normed vector space. If $F \subset E$ is closed and $K \subset E$ is compact. If $F \cap K = \emptyset$ prove that there exist $\varepsilon > 0$ such that $\text{dist}(F, K) := \inf\{\|x - y\|; \ x \in F \text{ and } y \in K\} > \varepsilon$.

1.5 – If M is a subspace of a normed space E, prove that $\overline{M}$ is also a subspace.

1.6 – Let (X, d) be a complete metric space. Consider a map $f : X \longrightarrow X$ and suppose that there exists $\lambda \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Consider $x_0 \in X$ and, for each $n \in \mathbb{N}$, define $x_n = f^n(x_0)$, where

$$f^n = \underbrace{f \circ \ldots \circ f}_{n \ times} \quad (n \in \mathbb{N}).$$

a) Show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X;

b) Show that $a := \lim_{n \to +\infty} x_n$ is the unique fix point of f.

2 Bounded Linear operators and Hahn-Banach Theorem

2.1 – Let E be a vector spaces and $\varphi : E \to \mathbb{R}$ a linear mapping. Suppose there exists $C > 0$ such that $\varphi(x) \leq C$ for all $x \in E$. Prove that $\varphi = 0$

2.2 – Let $\varphi : E \to F$ be a linear operator. If there exists $C > 0$ such that

$$\|\varphi(z)\| \leq C, \ \forall z \in E \text{ with } \|z\| < 1,$$

prove that $\|\varphi(x)\| \leq C, \forall x \in E$ such that $\|x\| = 1$. Conclude that $\varphi \in \mathcal{L}(E, F)$ and that $\sup_{\substack{x\in E \\ \|x\|<1}} |\varphi(x)| = \|\varphi\|$.

2.3 – Let E, F be vector spaces and $T : E \to F$ a bijective linear mapping. Prove that T^{-1} is linear.

2.4 – Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two normed vector space. Prove that the mapping

$$(x, y) \in E_1 \times E_2 \mapsto \|(x, y)\| = \|x\|_1 + \|y\|_2 \in \mathbb{R}$$

is a norm on $E_1 \times E_2$. And prove that $E_1 \times E_2$ is complete iff E_1 and E_2 are completes.

2.5 – Let $(E, \|\cdot\|)$ be a normed vector space.

a) Prove that the mappings

$$(x, y) \in E \times E \mapsto x + y \in E$$

and

$$(\lambda, x) \in \mathbb{R} \times E \mapsto \lambda x \in E$$

are continuous.

b) Fixed $x_0 \in E$ and $\lambda_0 \in \mathbb{R}$, $\lambda_0 \neq 0$. Prove that the mapping

$$x \in E \mapsto \lambda_0 x + x_0 \in E$$

is an homeomorphism.

2.6 – Let $T : E \longrightarrow F$ be an linear operator. Prove that $T \in \mathcal{L}(E, F)$ if, and only if, T maps bounded sets to bounded sets.

2.7 – Consider the space c_0 with its usual norm. For every element $u = (u_1, u_2, \ldots) \in c_0$ define

$$f(u) = \sum_{n=1}^{+\infty} \frac{1}{2^n} u_n.$$

Prove that f is continuous and find its norm.

2.8 – Let E be a infinite dimensional normed vector space. Recall that every vector space has a basis (Hamel basis). Construct a linear functional $f : E \longrightarrow \mathbb{R}$ that is not continuous.

2.9 – For each $n \in \mathbb{N}$, define

$$e_n = (0, \ldots, 0, \underbrace{1}_{n^{\text{th}}\ term}, 0, \ldots).$$

Prove that $(e_n)_{n=1}^{\infty}$ is not a Cauchy sequence in ℓ^p, where $p \in [1, +\infty)$ or $p = \infty$. In particular, conclude that those spaces are infinite dimensional.

2.10 – Let E be a normed vector space and suppose $x_1, \ldots, x_n \in E$ are linearly independent, where $n \in \mathbb{N}$. Prove that there exist $\varphi_1, \ldots, \varphi_n \in E'$ such that $\varphi_j(x_k) = \delta_{jk}$ for all $j, k \in \{1, \ldots, n\}$.

2.11 – Let E be a normed vector space. Prove that if $\varphi(u) = 0$, for all $\varphi \in E'$, then $u = 0$.

2.12 – Let E be a Banach space and $\widehat{\ } : E \to E''$ the canonical injection. Prove that if $X \subset E$ is closed, then $\widehat{X} \subset E''$ is closed. In particular, E is a closed subspace of E'', where we are identifying E with a subspace of E'' using the canonical injection.

2.13 – If M is a subspace of a normed vector space E, we set

$$M^{\perp} = \{\varphi \in E';\ \langle \varphi, x \rangle = 0,\ \forall x \in M.\}.$$

If M is a proper closed subspace of E, show that $M^{\perp} \neq \{0\}$.

2.14 – Let M be a proper closed subspace of a normed vector space E. Let $u_0 \in E$ such that $\text{dist}(u_0, M) = \inf_{v \in M} \|u_0 - v\| = d > 0$. Prove that there exists $\varphi \in E'$ such that $\varphi(u_0) = d$, $\varphi|_M = 0$ and $\|\varphi\|_{E'} = 1$.

2.15 – Let E be a normed space and let M be a closed subspace of E. We say $x \in E$ and $y \in E$ are **equivalents modulo** M if $x - y \in M$. In this case, we write

$$x = y (\text{mod } M).$$

a) Show that this is a equivalence relation. Let us denote by $[x]$ the equivalence class of each element $x \in E$;

b) Consider the **quotient space of** E **modulo** M, defined by

$$E/M = \{[x]; x \in E\}.$$

Show that the mappings

$$([x],[y]) \in E/M \times E/M \longmapsto [x+y] \in E/M$$

and

$$(\lambda,[x]) \in \mathbb{K} \times E/M \longmapsto [\lambda x] \in E/M$$

are well defined. Prove that E/M is a vector space with these operations.

c) Let F be a vector space and consider a linear mapping $T : E \longrightarrow F$. Show that there exists a linear bijection between

$$E/\ker T \text{ and } R(T).$$

3 Uniform Boundedness Principle, Open Mapping and Closed Graph theorem.

3.1 – Define a sequence of functionals $T_n : c_{00} \to \mathbb{R}$ by

$$T_n 0 = 0, \text{ and } T_n x = \sum_{i=1}^{n} x_i.$$

a) Prove that $T_n \in X'$.

b) Prove that for any $x \in X$, $(T_n x)$ is bounded, but (T_n) is not uniformly Bounded.

c) Why it doesn't contradict Banach-Steinhaus Theorem?

3.2 – Let E be a Banach space, F a normed vector space and $(T_n) \subset \mathcal{L}(E, F)$ such that $\sup_n \|T_n\| = +\infty$. Show that there exists an $x_0 \in E$ such that $\sup_n \|T_n x_0\| = +\infty$.

3.3 – Let E, F be Banach spaces and $T : E \longrightarrow F$ be an operator (not necessarily linear). Suppose there exist an operator $T^* : F' \longrightarrow E'$, called **adjoint operator**, defined by $T^*\varphi = \varphi \circ T$. Prove that

a) $T \in \mathcal{L}(E, F)$.

b) $\|T^*\| = \|T\|$.

(See a hint at the end.)

3.4 – Let E be a Banach space and F and G normed vector spaces. Suppose $u : E \times F \longrightarrow G$ is a separately continuous bilinear mapping, that is, for every $x \in E$ and $y \in F$, the linear mappings

$$u_x : w \in F \longmapsto u(x, w) \in G \text{ and } u_y : z \in E \longmapsto u(z, y) G$$

are continuous. Prove that u is continuous, that is, there exists a constant $C > 0$ such that

$$\|u(x, y)\| \leq C\|x\|\|y\|, \ \forall x \in E, \forall y \in F.$$

3.5 – Show that completeness of E is essential in the Banach-Steinhaus Theorem and cannot be omitted. Consider $x = (x_k) \in c_{00}$ and define $T_n x = n x_n$.

3.6 – Let E be a Banach space, F a normed space and $T_n \in \mathcal{L}(E, F)$ such that $(T_n x)$ is Cauchy in F for every $x \in E$. Show that $(\|T_n\|)$ is bounded.

* **3.7 –** Let $x = (x_n)$ be a real sequence such that $\sum_{n=1}^{\infty} x_n y_n$ converges for every $y = (y_n) \in c_0$. Show that $\sum_{n=1}^{\infty} |x_n|$ converges.

(See a hint at the end.)

3.8 – Let E be a normed vector space and M a closed subspace of E. For each $X \in E/M$, set

$$\|X\| = \inf\{\|x\|; x \in X\}.$$

a) Show that $[x] = x + M$ and that $\|[x]\| = \text{dist}(x, M)$ for each $x \in E$;

b) Show that the mapping defined in the statement of the exercise is a norm on E/M;

c) Show hat the canonical injection

$$\pi : x \in E \longmapsto [x] \in E/M$$

is continuous an open mapping. (See a hint at the end.)

(d)If E is a complete space, show that E/M is also complete.

3.9 – Show that every linear functional in E' is an open mapping.

3.10 – Let E, F be Banach spaces. If $T \in \mathcal{L}(E, F)$ is surjective, show that there exists $C > 0$ such that for every $y \in F$ the equation $Tx = y$ has a solution $x_y \in E$ such $\|x_y\| \le C\|y\|$.

3.11 – Let E be a Banach space and let $T : E \to E'$ be a linear operator satisfying

$$\langle Tx, y\rangle = \langle Ty, x\rangle, \ \forall x, y \in E.$$

Use Closed Graph Theorem to prove that T is bounded.

3.12 – Let E be a Banach space and $T : E \longrightarrow E'$ be a linear operator satisfying

$$\langle Tx, x\rangle \ge 0, \ \forall x \in E.$$

Prove that T is a bounded operator. (See a hint at the end.)

** **3.13 –** The aim of this exercise is prove the following proposition.

Proposition. *Every closed vector subspace of continuously differentiable functions in* $C^0([-1,1])$ *is finite dimensional.*

Part I

a) Let E be a closed subspace of in $C^0([-1,1])$, with the induced norm, that all its elements are continuously differentiable functions. Prove that E is a closed subspace in $C^1([-1,1])$ with the norm $\|\cdot\|_{C^1}$.

b) Prove $(E, \|\cdot\|_{C^1})$ is a Banach space.

c) Let us set $E_1 = (E, \|\cdot\|_{C^1})$ and $E_\infty = (E, \|\cdot\|_\infty)$. Prove that $I_d : E_1 \longrightarrow E_\infty$ is a homeomorphism.

Part II

In this part we will apply the *Ascoli-Arzelà Theorem*, namely

Theorem (Ascoli-Arzelà). *Let* K *be a compact metric space. A bounded subset* $\mathcal{F}$ *of* $C^0(K;\mathbb{R}^n)$ *is relatively compact if, and only if,* $\mathcal{F}$ *is equicontinuous, that is,*

$$\forall \varepsilon > 0,\ \exists \delta > 0 \textit{ such that } d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| \leq \varepsilon,\ \forall f \in \mathcal{F}.$$

a) Prove that $I_d(\overline{B_1})$ satisfies the hypotheses of Ascoli-Arzelà's theorem.

b) Conclude that $\overline{B_1}$ is compact and therefore $\dim E_1 < \infty$.

4 Weak Topologies

4.1 – Let E be a reflexive space and $(x_n)_{n\in\mathbb{N}} \subset E$ a sequence such that $(f(x_n))_{n\in\mathbb{N}}$ is convergent for all $f \in E'$. Show that there exists $x \in E$ such that $x_n \rightharpoonup x$ in $\sigma(E, E')$.

4.2 – Let $T \in \mathcal{L}(E, F)$. Show that if $x_n \rightharpoonup x$ in E, then $Tx_n \rightharpoonup Tx$ in F.

4.3 – Let E, F be Banach spaces. If $T : E \longrightarrow F$ is a linear operator that maps strongly convergent sequences to zero into weakly convergent sequences to zero, prove that T is continuous.

4.4 – Let E be a Banach space and let $A \subset E$ be a compact subset in the weak topology $\sigma(E, E')$. Prove that A is bounded.

4.5 – Let E be a reflexive Banach space and $K \subset E$ be a convex, closed and bounded set. Prove that K is compact in $\sigma(E, E')$.

4.6 – Let E be a Banach space and let $K \subset E$ be a compact set in the strong topology. If (x_n) is a sequence in K such that $x_n \rightharpoonup x$ in $\sigma(E, E')$, prove that $x_n \to x$ strongly.
(See a hint at the end.)

5 Hilbert Spaces

5.1 — Let E be a vector space endowed with a scalar product. If $u, v \in E$ and $u \perp v$, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

5.2 — Let H be a Hilbert space and $(x_n)_{n\in\mathbb{N}} \subset H$. Prove that if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$.

5.3 — Let H be a Hilbert space and let $(x_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in H. Prove the **Bessel's inequality**

$$\sum_{n=1}^{\infty} |(x_n, y)|^2 \leq \|y\|^2, \ \forall y \in H.$$

5.4 — Let H be a Hilbert space and (x_n) be a sequence in H. Prove that

$$x_n \to x \ \text{ in } \sigma(H, H') \Leftrightarrow (x_n, y) \to (x, y), \ \forall y \in H.$$

5.5 — Let H be a Hilbert space and $(x_n) \subset H$ an orthonormal sequence. Prove that $x_n \rightharpoonup 0$ in $\sigma(H, H')$.

5.6 — Let (e_n) be an orthonormal Hilbertian basis.

a) Given any sequence $(\alpha_n) \in \ell^2$. Prove that the series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges to some element $u \in H$ such that $(u, e_k) = \alpha_k$ for all $k \in \mathbb{N}$ and $\|u\|^2 = \sum_{k=1}^{\infty} \alpha_k^2$.

b) Prove that every separable Hilbert space is isomorphic to ℓ^2.

6 Compact operators, Spectral Theorem

6.1 – Let $f : X \to Y$ be a continuous function. If $A \subset X$, then $f(\overline{A}) \subset \overline{f(A)}$.

6.2 – Let X and Y be a normed vector spaces and let $f : X \to Y$ be a linear isometry. If $f(K) \subset Y$ is compact, then K is compact.

6.3 – Let X and Y be Banach spaces and let $T : X \longrightarrow Y$ be a linear, surjective isometry. If Y is reflexive, then X is reflexive.
(See a hint at the end.)

6.4 – Let H be a Hilbert space. Prove that if $T \in \mathcal{K}(H)$, then there exists a sequence $(T_n) \in \mathcal{K}(H)$ with finite rank such that $T_n \to T$.
(See a hint at the end.)

6.5 – Given $x = (x_1, x_2, \ldots) \in \ell^2$ define the operators

$$S_r x = (0, x_1, x_2, \ldots, x_{n-1}, \ldots) \text{ and } S_\ell x = (x_2, x_3, \ldots, x_{n+1}, \ldots),$$

respectively called the *right shift* and *left shift*.

a) Determine $\|S_r\|$ and $\|S_\ell\|$. Does they belong to $\mathcal{K}(\ell^2)$?

b) Prove that $EV(S_r) = \emptyset$.

c) Prove that $\sigma(S_r) = [-1, 1]$.

d) Prove that $EV(S_\ell) = (-1, 1)$. Determine the corresponding eigenspace.

e) Prove that $\sigma(S_\ell) = [-1, 1]$.

6.6 – Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous mapping and $E = (C^0([0,1]), \| \cdot \|_\infty)$. Define $J : E \longrightarrow E$ by

$$Jf(x) = \int_0^1 K(x, \xi) f(\xi) d\xi.$$

Show that $J \in \mathcal{K}(E)$. Does J has an inverse?

6.7 – Let $T : \ell^2 \longrightarrow \ell^2$ be a mapping defined by $Tx = \left(\frac{x_1}{2}, \frac{x_2}{2^2}, \ldots, \frac{x_n}{2^n}, \ldots\right)$, where $x = (x_n) \in \ell^2$.

a) Show that there exists a sequence (T_n) of finite-rank bounded linear operators such that $T_n \to T$.

b) Prove that $\sigma(T) = \{0\} \cup \left\{\frac{1}{2^n}; n \in \mathbb{N}\right\}$.

7 Answers and Hints

3.3 – Use Exercises 2.11 and 2.6.

3.7 – Define $T_n y = \sum_{k=1}^{n} x_n y_n$ and use Banach-Steinhaus Theorem.

3.8 – Verify that $\pi(B_E) = B_{E/M}$.

3.12 – Use Closed Graph Theorem: Given $(x_n) \subset E$ such that $x_n \to x$ in E and $Tx_n \to \varphi$ in E'. Passing to the limit in the inequality $\langle Tx_n - Ty, x_n - y\rangle \geq 0$ leads to

$$\langle \varphi - Ty, x - y\rangle \geq 0, \forall y \in E.$$

Choosing $y = x + tz$ with $t \in \mathbb{R}$ and $z \in E$, one sees that $\varphi = Tx$.

4.6 – Argue by contradiction.

6.3 – Prove that T^{**} is also a surjective isometry.

6.4 – Use projection.