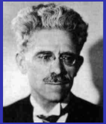
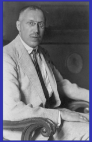
David Hilbert
1962-1943



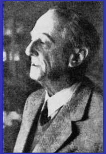
Maurice Fréchet
1878-1973



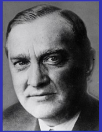
Hans Hahn
1879-1934

Functional Analysis

Frigyes Riez
1880-1956



Hugo Steinhaus
1887-1972



Stefan Banach
1892-1945

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Pós-Graduação em Matemática – PGMAT

Escola de Verão de 2021

Sumário

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Definition 1.1.

A *metric* *on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties for any $x, y, z \in X$:*

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0 \Leftrightarrow x = y$;
3. $d(x, y) = d(y, x)$;
4. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

A *metric space* *is the pair (X, d), where X is a set and d is a metric on X.*

Definition 1.2.

Let E *be a real vector space. A* *norm* *on* E *is a function* $\|\cdot\| : E \to \mathbb{R}$ *having the following properties for any* $x, y \in E$ *and any* $\lambda \in \mathbb{R}$*:*

1. $\|x\| \geq 0;$
2. $\|x\| = 0 \Leftrightarrow x = 0;$
3. $\|\lambda x\| = |\lambda| \|x\|;$
4. $\|x + y\| \leq \|x\| + \|y\|.$

A vector space E *endowed with a norm* $\|\cdot\|$ *is called* *normed vector space*[a] *and denoted by* $(E, \|\cdot\|)$.

[a] This definition was given (independently) by Banach, Hahn and Wiener.

Exercise 1.3.

1. Show that if $(E, \|\cdot\|)$ is a normed space, then E is a metric space with the metric defined by $d(x,y) = \|x-y\|$.
2. Examples of norms on $\mathbb{R}^n$ are:

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^2\right)^{\frac{1}{2}}, \; \|x\|_m = \max_{1\leq k\leq n} |x_k| \;\; \text{e} \;\; \|x\|_s = \sum_{k=1}^{n} |x_k|.$$

3. The set of all bounded function from X into $\mathbb{R}$, denoted by $\mathcal{B}(X)$, is a normed space with the norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

Exercise 1.4.

1. A sequence $(f_n)_{n\in\mathbb{N}}$ on $\mathcal{B}(X)$ converges to f on $\mathcal{B}(X)$ if, and only if, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on X.
2. Let $C^0([a,b])$ be the set of all continuous functions from $[a,b]$ into $\mathbb{R}$, which is a vector subspace of $\mathcal{B}([a,b])$. $C^0([a,b])$ is a normed vector space with the norm defined by:

$$\|f\|_{L^1} = \int_a^b |f(t)|dt.$$

Definition 1.5.

A complete normed vector space is called Banach space.

Example 1.6.

$\mathcal{B}(X)$ is a Banach space.

Exercise 1.7.

Show that $C^1([a,b])$, the set of all continuously differentiable functions $f:[a,b]\to\mathbb{R}$, with a norm defined by

$$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty,$$

is a Banach space.

Definition 1.8.

Let $1 \leq p < \infty$. We define ℓ^p to be the set of all $p-$summable real sequences, i.e.,

$$\ell^p = \left\{ (x_n)_{n\in\mathbb{N}};\ \sum_{n=1}^{\infty} |x_n|^p \text{ converges} . \right\}$$

with the norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} .$$

For $p = \infty$ we define $\ell^\infty := \mathcal{B}(\mathbb{N})$.

Proposition 1.9.

ℓ^p is a Banach space.

Exercise 1.10.

Assume $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Given $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$,

1. prove Hölder's Inequality (see [Dinamérico]):

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q \right)^{1/q}.$$

2. prove Minkowski's Inequality (see [Dinamérico]):

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^q \right)^{1/q}$$

3. Conclude ℓ^p is a normed space.

Exercise 1.11.

If M is a vector subspace of a normed vector space E, then M is also a normed vector space if we consider on M the same norm on E. In this case, the norm on M is said to be induced by the norm on E.

Example 1.12.

Let

- c: the set of all convergent sequences on $\mathbb{R}$.
- c_0: the set of all convergent sequences to zero.
- c_{00}: the set of all sequences which have only finitely many nonzero elements.

We have that $c_{00} \subset c_0 \subset c$ are normed vector spaces with the induced norm by ℓ^∞.

Proposition 1.13.

If M is a closed vector subspace of a Banach space E, then M is a Banach space with respect to the induced norm.

Exercise 1.14.

1. Show that $C^0(X)$, where X is compact, with the norm $\|\cdot\|_\infty$ is a Banach space.
2. The space c of example 1.12 is a Banach space.
3. Show that

$$x_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots, 0, \ldots\right)$$

is a Cauchy sequence in c_{00} and that $x_n \to x$, where

$$x = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right) \in c.$$

Conclude c_{00} is not a Banach space.

Definition 2.1.

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces. A mapping from E into F is called an operator. A mapping from E into its scalar field is called a functional.
We say a linear operator $T : E \to F$ is bounded [a] if there exists a constant $M > 0$ such that

$$\|Tx\|_F \leq M\|x\|_E, \ \forall x \in E.$$

We will denote by $\mathcal{L}(E, F)$ the vector space of all bounded linear operators form E into F.

[a] The word bounded is used here in the sense that T maps bounded sets into bounded sets. In the usual notion of a bounded function, no linear function could be bounded.

Proposition 2.2.

Let $T : E \to F$ be a linear operator. Then the following statements are equivalents:

1. T *is continuous in the origin.*
2. T *is bounded.*
3. T *is uniformly continuous.*

Exercise 2.3.

Prove that $\mathcal{L}(E,F)$ is a normed vector space with its norm defined by

$$\|T\| := \sup_{\substack{x\in E\\ x\neq 0}} \frac{\|Tx\|_F}{\|x\|_E}.$$

Proposition 2.4.

If $T \in \mathcal{L}(E,F)$, then

$$\|T\| = \sup_{\substack{x\in E\\ x\neq 0}} \frac{\|Tx\|_F}{\|x\|_E} = \sup_{\substack{x\in E\\ \|x\|\leq 1}} \|Tx\|_F = \sup_{\substack{x\in E\\ \|x\|=1}} \|Tx\|_F.$$

Proposition 2.5.

If E is a normed vector space and F is a Banach space, then $\mathcal{L}(E, F)$ is a Banach space.

Definition 2.6.

$\mathcal{L}(E, \mathbb{R})$ is called *topological dual* of E and is denoted by E'. Given $\varphi \in E'$ and $x \in E$ we shall often write $\langle \varphi, x \rangle$ instead $f(x)$. We say that $\langle\ ,\ \rangle$ is the *scalar product for the duality* E', E.

Corollary 2.7.

E' is a Banach space.

Definition 2.8.

Let E and F normed vector spaces. A linear operator $T : E \to F$ is said to be a topological isomorphism if T is bijective, T and T^{-1} are continuous. In this case we say E and F are isomorphic.

Proposition 2.9.

A surjective linear operator $T : E \to F$ is an isomorphism if, and only if, there exist constants $m, M > 0$ such that

$$m\|x\|_E \leq \|Tx\|_F \leq M\|x\|_E, \ \forall x \in E.$$

Exercise 2.10.

Let E, F be isomorphic vector spaces. E is a Banach space if, and only if, F is a Banach space.

Definition 2.11.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the same vector space E are said to be equivalents if the identity operator

$$Id_E : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

is an isomorphism.

Proposition 2.12.

In order to have two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ it is necessary and sufficient that there exist constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1, \ \forall x \in E.$$

Proposition 2.13.

Every finite dimensional normed spaces are isomorphic.

Exercise 2.14.

Let E be a finite dimensional normed vector space. Then

1. Every norms are equivalents.
2. E is a Banach space.
3. If $T : E \to F$ is linear then $T \in \mathcal{L}(E, F)$.
4. A subset of E is compact iff it is a closed and bounded set.

Proposition 2.15.

Let E be a normed vector space such that the closed unit ball is compact, then E is finite dimensional.

Lemma 2.16 (Riez).

Let F be a closed proper vector subspace of E. Then, given $0 < \varepsilon < 1$, there exists $x_0 \in E$ such that $\|x_0\| = 1$ and $\operatorname{dist}(x_0, F) \geq \varepsilon$.

Definition 3.1.

Let E be a real vector space. We say a functional $p : E \to R$ is sublinear if satisfies:

1. $p(x + y) \leq p(x) + p(y)$ *for every* $x, y \in E$. *(subadditive)*
2. $p(\lambda x) = \lambda p(x)$ *for every* $x \in E$ *and every* $\lambda > 0$. *(positive-homogeneous)*

Example 3.2.

If E is a normed vector space, a norm is a sublinear functional.

Theorem 3.3 (Teorema de Hahn-Banach Forma Analítica[1]).

Let E be a real vector space, M a subspace of E and $p : E \to \mathbb{R}$ a sublinear functional. If $\varphi : M \to \mathbb{R}$ is a linear functional that satisfies $\varphi(x) \leq p(x)$ for all $x \in M$, then ψ has a linear extension $\psi : E \to \mathbb{R}$. Furthermore, $\psi(x) \leq p(x)$ for all $x \in E$.

The theorem was discovered by H. Hahn (1927), rediscovered in its present form by S. Banach (1929) and generalized to complex vector spaces by H. F. Bohnenblust and A. Sobczyk (1938), see [Kreyszig, Chapter 4.2]

Remark 3.4.

The crux of Hahn-Banach's Theorem is not the existence of a linear extension of φ, but that this extension is still majorized by p.

Corollary 3.5 (Hahn-Banach Theorem for normed spaces).

Let M be a subspace of a normed vector space E. If $\varphi \in M'$, the there exists $\psi \in E'$ that extends φ and such that $\|\psi\|_{E'} = \|\varphi\|_{M'}$.

Corollary 3.6.

Let E be a normed vector space and $x_0 \in E \setminus \{0\}$. Then there exists $\psi \in E'$ such that $\langle \psi, x_0 \rangle = \|x_0\|$ and $\|\psi\|_{E'} = 1$.

Exercise 3.7.

Let E be a normed vector space. Prove that:

1. For every $x_0 \in E$ there exists $\varphi_0 \in E'$ such that $\|\varphi_0\| = \|x_0\|$ and $\langle \varphi_0, x_0 \rangle = \|x_0\|^2$.
2. For every $x \in E$, $\|x\| = \sup_{\substack{\varphi \in E' \\ \|\varphi\| \leq 1}} |\langle \varphi, x \rangle|$.

Definition 3.8.

Let $\mathcal{O}$ be a set with a partial order relation $\leq$. A subset $P \subset \mathcal{O}$ is said to be totally ordered if

$$\forall x, y \in P, \text{ one has } x \leq y \text{ or } y \leq x.$$

The set $\mathcal{O}$ is said to be inductive if every totally ordered subset P has an upper bound in $\mathcal{O}$, that is,

$$\exists c \in \mathcal{O} \text{ such that } x \leq c, \ \forall x \in P.$$

We say $m \in \mathcal{O}$ is a maximal element of $\mathcal{O}$ if there is no element $x \in \mathcal{O}$ such that $m \leq x$, except for $x = m$.

Lemma 3.9 (Zorn).

Every nonempty partially ordered set that is inductive has a maximal element.

Remark 3.10.

Zorn's lemma has many important applications in analysis. One of them is to prove that every vector space has a (Hamel) basis, see [Kreyszig, section 4.1]

Definition 3.11.

Let E be a normed vector space. The topological dual of E', denoted by E'' is called bidual of E.

Proposition 3.12.

For each $x \in E$, we define the evaluation functional $\widehat{x} : E' \to \mathbb{R}$ by $\widehat{x}(\varphi) = \langle \varphi, x \rangle$. Then $\widehat{x} \in E''$ and $\|\widehat{x}\| = \|x\|$. In other words, the operator $\widehat{\ } : E \to E''$ is an isometry.

Remark 3.13.

Some authors use the notation $J : E \to E''$ instead of $\widehat{\ }$.

Definition 3.14.

The isometry $\widehat{\ }$ *defined in Proposition 3.12 is called canonical embedding of* E *into* E''. *If the canonical embedding is surjective, that is,* $\widehat{E} = E''$, *we say* E *is reflexive.*

Exercise 3.15.

Show that:

1. Every reflexive space is Banach.
2. If $\dim E < \infty$ then E is reflexive.

Definition 3.16.

Let E be a normed vector space. A affine hyperplane of E is a subset of the form

$$H = \{x \in E;\ \varphi(x) = \alpha\},$$

where φ is a linear functional, not necessarily bounded, that does not vanish identically and $\alpha \in \mathbb{R}$. In this case, we denote H by $[\varphi = \alpha]$.

Proposition 3.17.

The hyperplane $[\varphi = \alpha]$ is closed if, and only if, $\varphi \in E'$.

Definition 3.18.

Let A and B be two subsets of the normed vector space E. We say that the hyperplane $[\varphi = \alpha]$ separates A and B if

$$\varphi(x) \leq \alpha, \ \forall x \in A \text{ and } \varphi(x) \geq \alpha, \ \forall x \in B.$$

Theorem 3.19 (1º Geometric form of the Hahn-Banach Theorem).

Let E be a normed vector space and $A, B \subset E$ nonempty disjoint convex subsets. If one of them is open, then there exists a closed hyperplane that separates A e B.

Definition 3.20.

Let A be an open convex subset of a normed vector space E such that $0 \in A$. The Minkowski Functional of A is a mapping $p_A : E \to \mathbb{R}$ defined by

$$p_A(x) = \inf\{\alpha > 0;\ x \in \alpha A\}.$$

Exercise 3.21.

Show that p_A is well defined, that is, for each $x \in E$, $X = \{\alpha > 0;\ x \in \alpha A\} \neq \emptyset$ and bounded from below.

Proposition 3.22.

The Minkowski functional satisfies:

- (a) $p_A(x) \geq 0, \ \forall x \in E$ *and* $p_A(0) = 0$.
- (b) p_A *is sublinear.*
- (c) $A = \{x \in E; \ p_A(x) < 1\}$.
- (d) *There exists* $M > 0$ *such that* $p_A(x) \leq M\|x\|, \forall x \in E$.

Lemma 3.23.

Let E *be a normed vector space and* $A \subset E$ *a nonempty open convex set. If* $x_0 \in E \setminus A$*, then there exists* $\varphi \in E'$ *such that* $\varphi(x_0) = \alpha$ *and* $\varphi(x) < \alpha, \forall x \in A$*. In particular, the hyperplane* $[\varphi = \alpha]$ *separates* $\{x_0\}$ *and* A*.*

Definition 3.24.

Let A *and* B *be two subsets of a normed vector space* E. *We say* $[\varphi = \alpha]$ *strictly separates* A *and* B *when there exists* $\varepsilon > 0$ *such that*

$$\varphi(x) \leq \alpha - \varepsilon,\ \forall x \in A \text{ and } \varphi(x) \geq \alpha + \varepsilon,\ \forall x \in B.$$

Theorem 3.25 (2º Geometric form of the Hahn-Banach theorem).

Let F *and* K *nonempty convex and disjoint subsets of a normed vector space E. If* F *is closed and* K *is compact, then there exists a closed hyperplane that strictly separates* F *and* K.

Remark 3.26.

1. *Assume $A, B \subset E$ are nonempty convex and disjoint sets. If we make no further assumption, it is in general impossible to to separate them by a closed hyperplane. One can even construct such an example in which A and B are both closed. See [Brezis, Exercise 1.14].*
2. *In finite-dimensional spaces one can always separate any two nonempty convex and disjoint sets. [Brezis, Exercise 1.9]*

Definition 3.27.

Let E be a normed vector space espaço vetorial normado and $F \subset E$ a vector subspace of E. The annihilator of F, denoted by $F^{\perp}$, is the set of all bounded linear operators that vanish in F, that is,

$$F^{\perp} = \{\varphi \in E'; \varphi(x) = 0, \ \forall x \in F\}$$

Corollary 3.28.

Given E a normed vector space and $F \subset E$ a subspace of E. If $F^{\perp} = \{0\}$, then F is dense in E, that is, $\overline{F} = E$.

Historical Note: [Lax, page 172]

Stefan Banach (1892-1945), a Polish mathematician, was one of the founding fathers of functional analysis. Banach spaces are named in recognition of his numerous and deep contributions, and for having written the first monograph on the subject (1932). He was the inspiration of the brilliant Polish school of functional analysis. During the Second World War, Banach was one of a group of people whose bodies were used by the Nazi occupiers of Poland to breed lice, in an attempt to extract an anti-typhoid serum. He died shortly after the conclusion of the war.

Teorema de Banach-Steinhaus

Theorem 4.1 (Baire).

Let (X, d) be a nonempty complete metric space. If X is a countable union of closed sets, then at least one of them has nonempty interior.

Theorem 4.2 (Banach-Steinhaus).

Let $(T_i)_{i\in I}$ be a family (not necessarily countable) in $\mathcal{L}(E, F)$, where E is a Banach space. If for each $x \in E$ there exists $C_x > 0$ such that

$$\|T_i x\| \leq C_x, \ \forall i \in I,$$

then there exists $C > 0$ such that

$$\|T_i\|_{\mathcal{L}(E,F)} < C, \ \forall i \in I.$$

Corollary 4.3.

Let (T_n) be a sequence in $\mathcal{L}(E,F)$ where E is a Banach space. If for each $x \in E$, $T_n x$ converges to an element of F, which will denote by Tx, then

1. $\sup_n \|T_n\|_{\mathcal{L}(E,F)} < \infty$,
2. $T \in \mathcal{L}(E,F)$,
3. $\|T\|_{\mathcal{L}(E,F)} \leq \liminf_{n\to\infty} \|T_n\|_{\mathcal{L}(E,F)}$.

Exercise 4.4.

1. Show that the condition E Banach is essential. Let c_{00} defined in example 1.12, which is not a Banach space, as we have seen in Example 1.14. Define $T_m : c_{00} \to \mathbb{R}$ by $T_m x = m x_m$, where $x = (x_n)_{n\in\mathbb{N}}$. $(T_m)_{m\in\mathbb{N}}$. Prove that $T_m x$ is bounded for each $x \in c_{00}$ but it is not uniformly bounded.
2. Let $S_\ell : \ell^2 \to \ell^2$ be the left shift operator, defined by $S_\ell(x_1, \dots, x_n, \dots) = (x_2, \dots, x_n, \dots)$, and define $T_n = S_\ell^n = \underbrace{S_\ell \circ \cdots \circ S_\ell}_{n \text{ times}}$. Find $\|T_n x\|$ and the limit operator of the Banach-Steinhaus Theorem.

Corollary 4.5.

Let E be a normed vector space and B be a subset of E. If for every $\varphi \in E'$ the set $\varphi(B)$ is bounded, then B is bounded.

Exercise 4.6.

Let E be a Banach space and B' be a subset of E'. If for every $x \in E$ the set $B'(x) = \{\varphi(x);\ \varphi \in B'\}$ is bounded then B' is bounded.

Theorem 5.1.

Let E and F be Banach spaces and $\varphi \in \mathcal{L}(E, F)$ surjective. Then there exists $c > 0$ such that

$$\varphi(B_E(0,1)) \supset B_F(0,c).$$

In particular, φ is an open mapping, that is, φ maps open sets of E into open sets of F.

Corollary 5.2.

Let E and F Banach spaces and $\varphi \in \mathcal{L}(E, F)$ bijective. Then $\varphi^{-1} \in \mathcal{L}(F, E)$.

Exercise 5.3.

Show that $T : \ell^1 \to \ell^1$ given by $T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, x_2/2, \ldots, x_n/n, \ldots)$ is linear, continuous and invertible, but its inverse is not an continuous operator.

Corollary 5.4.

Let E be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E such that E is a Banach space with respect to both norms. If there exists $C > 0$ such that

$$\|x\|_2 \leq C\|x\|_1, \ \forall x \in E,$$

then the norms are equivalents.

Example 5.5.

$C^0([0,1])$ with the norm $\|f\|_{L^1} = \int_0^1 |f(x)|dx$ is not a Banach space.

Exercise 5.6.

If $\varphi : E \longrightarrow F \in$ is an linear operator, prove that:

1. $\varphi(B_E(x,\varepsilon)) = \varphi(x) + \varepsilon\varphi(B_E(0,1))$.
2. $r\varphi(B_E(0,1)) = \varphi(B_E(0,r))$, $\forall r > 0$.
3. $B(x,r) \subset kA,\ k > 0 \Rightarrow B\left(\frac{x}{k}, \frac{r}{k}\right) \subset A$.

Closed Graph Theorem

Definition 5.7.

A linear operator $T : E \to F$ is said to be closed if its graph

$$G(T) = \{(x, Tx) \subset E \times F;\ x \in E\}$$

is a closed subspace of $E \times F$ with respect to the norm

$$\|(x, y)\|_{E\times F} = \|x\|_E + \|y\|_F.$$

Equivalently, if for any sequence $(x_n) \subset E$ such that $x_n \to x$ in E and $Tx_n \to y$ in F one has that $Tx = y$.

Exercise 5.8.

Every linear continuous operator is closed.

Example 5.9.

The operator $T^{-1} : Im(T) \subset \ell^1 \to \ell^1$ of exercise 5.3 is closed but is not continuous.

Theorem 5.10 (Closed Graph Theorem).

Let E, F Banach spaces. If $\varphi : E \to F$ is a closed linear operator, then φ is continuous.

Example 5.11.

$C^1([0,1])$ with respect to the norm $\|\cdot\|_\infty$ is not a Banach space.

Basic Topological Concepts

Definition 6.1.

A *topology* *on a set* X *is a collection* $\mathcal{T}$ *of subsets of* X*, called* *opens sets* *of* X*, such that* $\emptyset$ *and* X *are open and such that arbitrary unions and finite intersections of open sets are open.*

Definition 6.2.

Suppose $\mathcal{T}$ *e* $\mathcal{T}'$ *are two topologies on a given set* X*. We say* $\mathcal{T}$ *is* *coarser* *than* $\mathcal{T}'$ *whenever* $\mathcal{T} \subset \mathcal{T}'$*. We also say* $\mathcal{T}'$ *is* *finer* *than* $\mathcal{T}$*.*

Definition 6.3.

Let X be a nonempty topological space. A neighborhood of a point $x \in X$ is an open set that contains x.
A neighborhood basis for a point $x \in X$ is a set $\mathcal{B}(x)$ which elements are neighborhood of x and such that any neighborhood of x contain an element of $\mathcal{B}(x)$, that is, if V is a neighborhood of x, then there exists $\mathcal{U} \in \mathcal{B}(x)$ such that $\mathcal{U} \subset V$.

Example 6.4.

Let M be a metric space and consider $x \in M$. Then the set $\{B(x, r);\ r > 0\}$ is a neighborhood basis for x.

Definition 6.5.

A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Definition 6.6.

A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct point of X, there exist disjoint neighborhoods U_1 and U_2 of x_1 and x_2, respectively.

Definition 6.7.

Let X be a topological space. We say a sequence $(x_n) \subset X$ converge to $x \in X$ if for every neighborhood U of x there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.

Initial topology

Definition 6.8.

Let X be a nonempty set (without any structure), $(Y_i)_{i\in I}$ a collection of topological spaces and $\mathcal{F} = (\varphi_i)_{i\in I}$ a family of mappings $\varphi_i : X \to Y_i$. The initial topology on X, denoted by $\sigma(X, \mathcal{F})$, is that generated by the subbasis

$$\mathcal{S} = \{\varphi_i^{-1}(\omega_i);\ \omega \subset Y_i \text{ open},\ i \in I\}$$

Proposition 6.9.

1. $\sigma(X, \mathcal{F})$ *is the coarsest topology that make the functions in* $\mathcal{F}$ *continuous.*
2. *For each* $x \in X$, *the set*

$$\mathcal{B}(x) = \left\{ \bigcap_{j \in J} \varphi_j^{-1}(\omega_j);\ J \subset I \text{ is finite and } \omega_j \text{ is a neighborhood of } \varphi_j(x) \text{ in } Y_j. \right\}$$

is a neighborhood basis for x *with respect to* $\sigma(X, \mathcal{F})$.

Exercise 6.10.

A classical example of this kind of topology is the product topology on the product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ of a family of sets $(X_\alpha)_{\alpha\in\Lambda}$. This is the coarser topology that makes all projection mappings $\pi_\alpha : X \to X_\alpha$ continuous. Review the product topology and Tychonoff Theorem in [Munkres, Sections 2-19 and 5-37].

Proposition 6.11.

Let X be a set with the topology $\sigma(X,\mathcal{F})$ and (x_n) be a sequence in X. Then,

$$x_n \to x \text{ in } X \Leftrightarrow \varphi_i(x_n) \to \varphi_i(x),\ \forall i \in I.$$

Proposition 6.12.

Let Z be a topological space and let X with the topology $\sigma(X,\mathcal{F})$. A mapping $\psi : Z \longrightarrow X$ is continuous iff $\varphi_i \circ \psi : Z \longrightarrow Y_i$ is continuous for all $i \in I$.

Definition 6.13.

If E is a normed vector space we call $\sigma(E,E')$ the weak topologoy on E.

Properties

From Proposition 6.9 we have the following properties to $\sigma(E,E')$:

1. It the coarsest topology that makes the functions in E' continuous.
2. For each $x \in E$, a neighborhood basis for x is given by

$$\mathcal{B}(x) = \left\{ \bigcap_{i\in I} f_i^{-1}(\omega_i);\ f_i \in E',\ I \text{ is finite and } \omega_i \text{ is a neighborhood of } f_i(x) \right\}$$

.

Proposition 6.14.

The weak topology $\sigma(E, E')$ is Hausdorff.

Proposition 6.15.

Let E be a normed vector space and $x_0 \in E$. The following family of sets

$$V(x_0, f_1, \ldots, f_k; \varepsilon) = \{x \in E;\ |f_i(x) - f_i(x_0)| < \varepsilon,\ \forall i = 1, 2, \ldots, k\}$$
$$= \bigcap_{i=1}^{k} f_i^{-1}\Big((f_i(x_0) - \varepsilon, f_i(x_0) + \varepsilon)\Big)$$

form a neighborhood basis of x_0 for the weak topology $\sigma(E, E')$.

Proposition 6.16.

If E is a finite dimensional space, then the weak and strong topologies are the same.

Proposition 6.17.

Let E be a normed vector space and $C \subset E$ a convex subset. Then C is closed in the weak topology if, and only if, it is closed in the strong topology.

The weak topology is strictly coarser than the strong topology

Proposition 6.18.

If E is a infinite dimensional normed vector space then any neighborhood V of x_0 in $\sigma(E, E')$ contain a line passing through x_0.

Example 6.19.

1. If E is a infinite dimensional normed vector space then the unit sphere $S = \{x \in E;\ \|x\| = 1\}$ is not closed in the weak topology $\sigma(E, E')$. Precisely, the closure of S in the weak topology equals to the closed unit ball in E.
2. If E is a infinite dimensional normed vector space then the unit ball is not weakly open.

Weak convergence

Definition 6.20.

Given a sequence $(x_n) \subset E$, we denote by $x_n \rightharpoonup x$ or more precisely

$$x_n \rightharpoonup x \text{ weakly in } \sigma(E, E'),$$

the convergence of (x_n) to x in the weak topology $\sigma(E, E')$.

Proposition 6.21.

Let (x_n) be a sequence in E. Then,

1. $x_n \rightharpoonup x$ *if, and only if,* $\langle f, x_n \rangle \to \langle f, x \rangle$ *for all* $f \in E'$.
2. *If* $x_n \to x$ *strong, then* $x_n \rightharpoonup x$ *weak in* $\sigma(E, E')$.
3. *If* $x_n \rightharpoonup x$ *weak in* $\sigma(E, E')$, *then* $(\|x_n\|)$ *is bounded and* $\|x\| \leq \liminf \|x_n\|$
4. *If* $x_n \rightharpoonup x$ *weak in* $\sigma(E, E')$ *and* $f_n \to f$ *strong in* E', *then* $\langle f_n, x_n \rangle \to \langle f, x \rangle$.

Proposition 6.22.

Let E and F be Banach spaces and $T : E \longrightarrow F$ be linear operator. $T \in \mathcal{L}(E, F)$ if, and only if, T is continuous from E into F with its respective weak topologies.

Weak* topology

Definition 6.23.

Let E be a normed vector space. The weak topology on E' is the topology $\sigma(E', \widehat{E})$, that is, the coarsest topology on E', that make the family of linear functionals $(\widehat{x})_{x\in E}$ continuous. Since we can identify E and $\widehat{E}$, we will denote this topology simply by $\sigma(E', E)$.*

Remark 6.24.

1. *Since $\widehat{E} \subset E''$, $\sigma(E', E)$ is coarser than $\sigma(E', E'')$, that is, the weak topology on E'. In other words, the weak* topology has fewer open set than the weak topology on E'.*
2. *Weak topologies are important because a coarser topology has more compact sets. For example, the closed unit ball in E', which is never compact in the strong topology (unless $\dim E < +\infty$ as we have seen in Proposition 2.15) is always compact in the weak* topology, as we will see in Theorem 6.32. Hence, this kind of topology has applications on minimization problems, for example.*

Proposition 6.25.

1. *The weak* topology is generated by the subbasis*

$$\{\widehat{x}^{-1}(\omega_x);\ \omega_x \text{ is an open set in } \mathbb{R},\ \text{and } x \in E\}$$

2. *For each $f \in E'$, a neighborhood of f for the topology $\sigma(E', E)$ is given by*

$$\mathcal{B}(f) = \left\{ \bigcap_{x \in I} \widehat{x}^{-1}(\omega_x);\ I \subset E \text{ finite and } \omega_x \text{ is a neighborhood of } \langle \widehat{x}, f \rangle \right\}.$$

Proposition 6.26.

The weak topology $\sigma(E', E)$ is Hausdorff.*

Proposition 6.27.

Let E be a normed vector space and $f_0 \in E'$. The following family of sets

$$V(f_0; x_1, \ldots, x_k; \varepsilon) = \{f \in E';\ |\langle f, x_i\rangle - \langle f_0, x_i\rangle| < \varepsilon,\ \forall i = 1, 2, \ldots, k\}$$
$$= \bigcap_{i=1}^{k} \widehat{x}_i^{-1}\Big((\langle \widehat{x}_i, f_0\rangle - \varepsilon, \langle \widehat{x}_i, f_0\rangle + \varepsilon)\Big)$$

form a neighborhood basis of f_0 for the weak topology $\sigma(E', E)$.*

Definition 6.28.

Let $(f_n) \subset E'$ *be a sequence, we will denote by* $f_n \overset{*}{\rightharpoonup} f$ *or more explicitly by*

$$f_n \overset{*}{\rightharpoonup} f \text{ weakly in } \sigma(E', E),$$

the convergence of (f_n) *to* f *in the topology* $\sigma(E', E)$.

Proposition 6.29.

Let (f_n) *be a sequence in* E'. *Then,*

1. $f_n \overset{*}{\rightharpoonup} f$ *if, and only if,* $\langle f_n, x\rangle \to \langle f, x\rangle$ *for all* $x \in E$.
2. *If* $f_n \to f$ *strong, then* $f_n \rightharpoonup f$ *weak in* $\sigma(E', E'')$. *If* $f_n \rightharpoonup f$ *weak in* $\sigma(E', E'')$, *then* $f_n \overset{*}{\rightharpoonup} f$ *in* $\sigma(E', E)$
3. *if* E *is Banach and* $f_n \overset{*}{\rightharpoonup} f$ *weak in* $\sigma(E', E)$, *then* $(\|f_n\|)$ *is bounded and* $\|f\| \le \liminf \|f_n\|$
4. *If* E *is Banach and* $f_n \overset{*}{\rightharpoonup} f$, $x_n \to x$ *strong in* E, *then* $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

Exercise 6.30.

Prove Proposition 6.29.

Remark 6.31.

When E is finite-dimensional the three topologies on E' coincide, since $\widehat{E} = E''$ and hence $\dim E = \dim E''$.

Banach-Alaoglu-Bourbaki Theorem

Theorem 6.32 (Banach-Alaoglu-Bourbaki).

Let E be a normed vector space. Then the closed unity ball in E', that is,

$$B_{E'} = \{f \in E'; \|f\|_{E'} \leq 1\},$$

is compact in the weak topology $\sigma(E', E)$.*

Kakutani's Theorem

Theorem 6.33 (Kakutani).

Let E be a Banach space. Then E is reflexive if and only if

$$B_E = \{x \in E;\ \|x\| \leq 1\}$$

is compact in the weak topology $\sigma(E, E')$.

Lemma 6.1 (Goldstine).

Let E be any Banach space. Then $\widehat{B}_E$ is dense in $B_{E''}$ with respect to the weak topology $\sigma(E'', E')$, that is, $\overline{\widehat{B}_E}^{\sigma(E'',E')} = B_{E''}$.*

Exercise 6.34.

See proof of Goldstine's Lemma in [Brezis, Lemma 3.4].

Definition 7.1.

Let E be a vector space. A scalar product on E is a bilinear form that is symmetric positive definite, that is, a mapping $(\cdot,\cdot) : E \times E \to \mathbb{R}$ that satisfies the following properties:

1. $(u + \lambda v, w) = (u, w) + \lambda(v, w)$, $\forall u, v \in E$ *and* $\lambda \in \mathbb{R}$.
2. $(u, v) = (v, u)$, $\forall u, v \in E$.
3. $(u, u) > 0$, *for all* $u \neq 0$.

Example 7.2.

Examples of scalar product:

1. $(x, y) = \sum_{i=1}^{n} x_i y_i$, $x, y \in \mathbb{R}^n$.
2. $(x, y) = \sum_{n=1}^{\infty} x_n y_n$, $x, y \in \ell^2$.
3. $(f, g) = \int_a^b f(x)g(x)dx$, where $f, g \in C^0([a, b])$.

Proposition 7.3 (Cauchy-Schwartz inequality).

Let E be a space with an scalar product and define $\|u\| = \sqrt{(u, u)}$. Then

$$|(u, v)| \leq \|u\|\|v\|, \text{ para todos } u, v \in E.$$

Exercise 7.4.

Let E be a space with an scalar product. Prove that:

1. The mapping $u \in E \longmapsto \|u\| \in \mathbb{R}$ is a norma on E. This norm is called norm induced by the scalar product.
2. The Parallelogram Law holds

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2), \ \forall u, v \in E.$$

Definition 7.5.

A Hilbert space is a vector space endowed with a scalar product such that H is complete for the norm induced by the scalar product.

Exercise 7.6.

1. Show that if E is a vector space endowed with a scalar product, then the polar identity holds

$$(u, v) = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right)$$

2. Show that if E is a normed vector space, than its norm is induced by a scalar product if, and only if, it satisfies the parallelogram law.

Definition 7.7.

Let E be a space endowed with a scalar product. We say $u, v \in E$ are orthogonal, and we denote by $u \perp v$, if $(u, v) = 0$. If M is a subset of E, we denote by $M^{\perp}$ the set of all vector that are orthogonal to all elements of M, that is,

$$M^{\perp} = \{u \in E;\ (u, v) = 0,\ \forall v \in M\}.$$

Exercise 7.8.

Show that

1. If $u \perp v$, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.
2. If M is a closed subspace of E, then $M^\perp$ is closed.
3. $M \cap M^\perp = \{0\}$

Proposition 7.9.

In a vector space endowed with a scalar product, $u \perp v$ if, and only if,

$$\|u+tv\| \geq \|u\|, \ \forall t \in \mathbb{R}.$$

Definition 7.10.

A vector space E is said to be the direct sum of two subspaces E_1,E_2 written $E = E_1 \oplus E_2$, if each $u \in E$ has a unique representation

$$u = u_1 + u_2,\ u_1 \in E_1,\ u_2 \in E_2.$$

Exercise 7.11.

$E = E_1 \oplus E_2$ if, and only if, $E = E_1 + E_2$ and $E_1 \cap E_2 = \{0\}$.

Theorem 7.12 (Orthogonal Projection).

Let E be any closed subspace of a Hilbert space H. Then

$$H = E \oplus E^{\perp}.$$

Lemma 7.13.

Let E be any closed subspace of a Hilbert space H. Then for every $v \in H$ there exists a unique $u \in E$ such that

$$\operatorname{dist}(v, E) := \inf_{w \in E} \|v - w\| = \|v - u\|.$$

Definition 7.14.

Let E be any closed subspace of a Hilbert space H. The Orthogonal Projection is the operator $P_E : H \longrightarrow E$ given by $P_E v = v_1$, where $v = v_1 + v_2$ is given by the direct sum $H = E \oplus E^{\perp}$.

Remark 7.15.

It follows immediately from the former proof:

1. $\|v - P_E v\| = \mathrm{dist}(v, E) = \inf_{w \in E} \|v - w\|$.
2. $P_E u = u$, *for all* $u \in E$, *that is,* $(P_E)|_E = I_d$.
3. $v - P_E v \in E^{\perp}$, *for all* $v \in H$.

Corollary 7.16.

Let E be any closed subspace of a Hilbert space H. Then $E = (E^{\perp})^{\perp}$.

Exercise 7.17.

Verify the following properties of orthogonal projection:

1. $P_E \in \mathcal{L}(H, E)$ with $\|P_E\| = 1$.
2. P_E is surjective .
3. $P_E^2 = P_E$ (idempotent) and $(P_E)|_E = I_d$.
4. $E^\perp = \ker(P_E)$ and $P_{E^\perp} = I_d - P_E$.
5. $P_E \cdot P_{E^\perp} = P_{E^\perp} \cdot P_E = 0$.

Theorem 7.18 (Riez-Fréchet Representation Theorem).

Let H be a Hilbert space. For every $\varphi \in H'$ there exists a unique $u \in H$ such that

$$\langle \varphi, v \rangle = (u, v), \forall v \in H.$$

Moreover, $\|u\| = \|\varphi\|_{H'}$.

Corollary 7.19.

Let H be a Hilbert space. Then, the natural map $\gamma : H \longrightarrow H'$, defined by $\gamma(u) = (u, \cdot)_H$, is a surjective linear isometry. In particular, for any $\varphi \in H'$ and $v \in H$ we have that

$$(\gamma^{-1}(\varphi), v) = \langle \varphi, v \rangle$$

Corollary 7.20.

Every Hilbert space is reflexive.

Definition 7.21.

Let H be a Hilbert space. A bilinear form $b : H \times H \to \mathbb{R}$ is said to be

1. *continuous if there is a constant $C > 0$ such that*

$$|b(u, v)| \leq C\|u\|\|v\|, \ \forall u, v \in H.$$

2. *coercive if there is a constant $\alpha > 0$ such that*

$$b(u, u) \geq \alpha\|u\|^2, \ \forall u \in H.$$

Proposition 7.22.

Let H be a Hilbert space and assume $b : H \times H \to \mathbb{R}$ is a continuous bilinear form. Then, there exists a unique operator $T_b \in \mathcal{L}(H)$ such that

$$b(u, v) = (T_b u, v), \ \forall v \in H.$$

Theorem 7.23 (Lax-Milgram).

Let H be a Hilbert space and assume $b : H \times H \to \mathbb{R}$ is a continuous coercive bilinear form. Then, given any $\varphi \in H'$ there exists a unique $u \in H$ such that

$$\langle \varphi, v \rangle = b(u, v), \ \forall v \in H.$$

Definition 7.24.

Let $(E_n)_{n\in\mathbb{N}}$ *be a sequence of closed subspaces of* H. *One says that* H *is the* *Hilbert sum* *of the* $E_n's$ *and one writes* $H = \oplus_n E_n$ *if*

1. *the spaces* E_n *are mutually orthogonal, i.e.,*

$$(u, v) = 0, \ \forall u \in E_n, \ \forall v \in E_m, \ m \neq n.$$

2. *the linear space spanned by* $\cup_{n=1}^{\infty} E_n$ *is dense in* H.

Theorem 7.25.

Assume $H = \oplus_n E_n$. *Given* $u \in H$, *set*

$$u_n = P_{E_n} u \text{ and } S_n = \sum_{k=1}^{n} u_k.$$

Then, we have $\lim_{n \to +\infty} S_n = u$ *and*

$$\sum_{k=1}^{+\infty} |u_k|^2 = |u|^2 \text{ (Bessel-Parseval's identity)}$$

Lemma 7.26.

Assume that (v_n) *is any sequence in* H *such that*

$$(v_n, v_m) = 0, \ \forall m \neq n \text{ and } \sum_{k=1}^{\infty} \|v_k\|^2 < +\infty.$$

Set $S_n = \sum_{k=1}^{n} v_k$, *then*

$$\lim_{n\to\infty} S_n = S \in H \text{ and } \|S\|^2 = \sum_{k=1}^{\infty} \|v_k\|^2.$$

Definition 7.27.

A sequence $(e_n)_{n\in\mathbb{N}}$ *in a Hilbert space* H *is said to be an orthonormal basis or a Hilbert basis of* H *if it satisfies the following properties:*

1. $|e_n| = 1$ *for all* $n \in \mathbb{N}$ *and* $(e_n, e_m) = 0, \ \forall m \neq n$,
2. *the linear space spanned by the* e_n*'s is dense in* H.

Not to be confused with an Hamel basis, which is a family in H such that every element in H can be uniquely written as a finite linear combination of the basis's element.

Corollary 7.28.

Let (e_n) *be an orthonormal basis. Then for every* $u \in H$*, we have*

$$u = \sum_{k=1}^{\infty} (u, e_k) e_k \text{ and } \|u\|^2 = \sum_{k=1}^{\infty} |(u, e_k)|^2.$$

Conversely, given any sequence $(\alpha_n) \in \ell^2$*, the series* $\sum_{k=1}^{\infty} \alpha_k e_k$ *converges to some element* $u \in H$ *such that* $(u, e_k) = \alpha_k$ *for all* $k \in \mathbb{N}$ *and* $\|u\|^2 = \sum_{k=1}^{\infty} \alpha_k^2$*.*

Definition 7.29.

We say that a metric space E is separable if there exists a subset $D \subset E$ that is countable and dense.

Exercise 7.30.

Let E be a separable metric space and let $F \subset E$ be any subset. Then F is also separable.

Theorem 7.31.

Every separable Hilbert space has an orthonormal basis.

Exercise 7.32.

See the proof on [Brezis, Theorem 5.11].

Preliminaries: Compact Metric Spaces

Definition 8.1.

A metric space M is said to be compact if every open covering of M contains a finite subcollection that also covers M.

Definition 8.2.

A subset X of a metric space M is said to be relatively compact ou precompact, if its closure $\overline{X}$ is compact. X is said to be totally bounded if for every $\varepsilon > 0$, we can write

$$X = \bigcup_{n=1}^{n} B(x_i, \varepsilon),$$ *where $B(x_i, \varepsilon)$ are open balls in M.*

Exercise 8.3.

Let X be a subset of a metric space M. If X is totally bounded, show that, for every $\varepsilon > 0$, X is contained in a finite union of open balls of radius $\varepsilon > 0$ with center in points of X.

Proposition 8.4.

If M is a metric space, then the following sentences are equivalents:

1. *M is compact.*
2. *Every infinite subset of M has an accumulation point.*
3. *Every sequence in M has a convergent subsequence.*
4. *M is complete and totally bounded.*

Exercise 8.5.

1. Every totally bounded set in a complete metric space is relatively compact.
2. Let M an N be metric spaces. If $X \subset M$ is compact and $F : M \to N$ is continuous, then $F(X)$ is compact.

Adjoint Operator

Definition 8.6.

Let E, F *be normed vector spaces and let* $A \in \mathcal{L}(E, F)$. *The* *adjoint operator* *of* A *is the operator* $A^* : F' \longrightarrow E'$ *defined by*

$$\langle A^*\varphi, v\rangle = \langle \varphi, Av\rangle, \ \forall \varphi \in F' \text{ and } v \in E.$$

Proposition 8.7.

The adjoint operator is well defined and $\|A^*\| = \|A\|$.

Remark 8.8.

It is possible to define an adjoint operator for unbounded linear operators, since its domains is densely defined. See [Brezis, Section 2.6].

Definition 8.9.

Let E, F be normed vector space. We say that a linear operator $T : E \longrightarrow F$ is compact if T maps bounded sets in E into relatively compact sets in F. We denote by $\mathcal{K}(E, F)$ the set of all compact operators in $\mathcal{L}(E, F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E, E)$.

Exercise 8.10.

1. $T : E \to F$ is compact iff for every bounded sequence (x_n) in E, (Tx_n) has a convergent subsequence.
2. $T : E \to F$ is compact iff for every sequence $(x_n) \subset B_E$, (Tx_n) has a convergent subsequence.
3. $\mathcal{K}(E, F)$ is a vector subspace of $\mathcal{L}(E, F)$.

Proposition 8.11.

Let F be a Banach space. Then $\mathcal{K}(E,F)$ is a closed subspace of $\mathcal{L}(E,F)$. In particular, $\mathcal{K}(E,F)$ is Banach.

Definition 8.12.

Let E, F be normed vector space, a linear operator $T : E \longrightarrow F$ is said to be finite rank if its image $R(T)$ is a finite-dimensional subspace of F. Clearly, any finite rank bounded linear operators are compact operators.

Corollary 8.13.

Let F be a Banach space. If (T_n) is a sequence of finite-rank bounded linear operator such that $T_n \to T$ em $\mathcal{L}(E,F)$, then T is a compact operator.

Proposition 8.14.

Let E, F and G three normed vector spaces. Let $T \in \mathcal{L}(E, F)$ and $S \in \mathcal{L}(F, G)$. If T or S is compact, then $S \circ T$ is compact.

Theorem 8.15 (Schauder).

Let E, F be normed vector spaces. $T : E \to F$ is compact iff $T^* : F' \longrightarrow E'$ is compact.

Theorem 8.16 (Fredholm Alternative).

Let E be a Banach Space and $T \in \mathcal{K}(E)$. Then

1. $\ker(I-T)$ *is finite dimensional.*
2. $R(I-T)$ *is closed, and more precisely* $R(I-T) = \ker(I-T^*)^{\perp}$.
3. $R(I-T) = E$ *iff* $\ker(I-T) = \{0\}$.
4. $\dim \ker(I-T) = \dim \ker(I-T^*)$

Exercise 8.17.

See proof of Fredholm Alternative in [Brezis, Theorem 6.6]

Definition 8.18.

Let E be a normed vector space and $T \in \mathcal{L}(E)$. The resolvent set, denoted by $\rho(T)$, is defined by

$$\rho(T) = \{\lambda \in \mathbb{R};\ (T - \lambda I) \text{ is bijective from } E \text{ onto } E\}.$$

The spectrum, denoted by $\sigma(T)$, is the complement of the resolvent set, i.e., $\sigma(T) = \mathbb{R} \setminus \rho(T)$. A real number λ is said to be an eigenvalue of T if $\ker(T - \lambda I) \neq \{0\}$ and $\ker(T - \lambda I)$ is the correspondent eigenspace. The set of all eigenvalues is denoted by $EV(T)$.

Remark 8.19.

If E is a Banach space and $\lambda \in \rho(T)$ then, from Corollary 5.2 $(T - \lambda I)^{-1} \in \mathcal{L}(E)$.

Proposition 8.20.

Let E be a Banach space and $T \in \mathcal{L}(E)$. Then the spectrum $\sigma(T)$ is compact and $\sigma(T) \subset [-\|T\|, \|T\|]$.

Theorem 8.21.

Let E be a Banach space and $T \in \mathcal{K}(E)$. Then one has:

1. *If $\dim E = +\infty$, then $0 \in \sigma(T)$.*
2. $\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$.
3. 0 *is the only possible point of accumulation of $EV(T)$.*
4. $EV(T)$ *is countable.*

Self-Adjoint Operators

If H is a Hilbert space and $T \in \mathcal{L}(H)$, we can identify H' and H and see T^* as a bounded operator from H into itself.

Definition 8.22.

Let H be an Hilbert Space. A operator $T \in \mathcal{L}(H)$ is said to be self-adjoint if $T^* = T$, i.e.,

$$(Tu, v) = (u, Tv), \ \forall u, v \in H.$$

Proposition 8.23.

Let $T \in \mathcal{L}(H)$ be a self-adjoint operator. Set

$$m = \inf_{\substack{u \in H \\ \|u\|=1}} (Tu, u) \text{ and } M = \sup_{\substack{u \in H \\ \|u\|=1}} (Tu, u).$$

Then $\sigma(T) \subset [m, M]$, $m \in \sigma(T)$ and $M \in \sigma(T)$. Moreover, $\|T\| = \max\{|m|, |M|\}$.

Corollary 8.24.

If T is a self-adjoint operator such that $\sigma(T) = \{0\}$, then $T = 0$.

Theorem 8.25 (Spectral Theorem for Compact Self-adjoint Operators).

Let H be a separable Hilbert space. If T is a compact self-adjoint operator, then there exists a Hilbert basis composed of eigenvalues of T.